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Some Remarks Concerning the Question of Localization of Elementary Particles

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Abstract

In this paper we give an elementary discussion of the localization concept of relativistic particles. We prove the so-called Hegerfeldt theorem, which says that localization of a one-particle state in a finite space(-time) volume is inconsistent with causality, in two different ways. The proofs are elementary and use on the one hand the same type of arguments as used in the proof of the well-known theorem of Reeh-Schlieder and on the other a remark due to Borchers.

1. *Introduction*

It is well known that a careful analysis of the inhomogenous Lorentz group (Wigner, 1939) has given rise to a very useful classification of relativistic particles. The corresponding relativistic particle states are then supposed to transform according to irreducible representations of this fundamental group, if one introduces the restrictions

$$m \geqq 0 \tag{1.1}$$

$$E \geqq 0 \tag{1.2}$$

where m is the particle mass and

$$E \equiv (m^2 + \mathrm{p}^2)^{1/2} \tag{1.3}$$

is the corresponding relativistic energy.

On the empirical level of elementary particle physics we have, however, also a notion of "localizability" of particles. In nonrelativistic quantum mechanics, e.g., this concept is built into the theory on a very fundamental level[1]. It is associated with the "canonical" quantization as expressed by the commutation relation

$$[x, p] \subset i\hbar \tag{1.4}$$

where x is the nonrelativistic position operator and p the corresponding generator of translations in space. This commutation relation emerges in a very natural way when one introduces *transitive systems of imprimitivities* (in the sense of Mackey, 1968) as a representation of the fact that the particle can be localized in the physical space (Jauch, 1968). Indeed, this concept is so fundamental that an irreducible system of imprimitivities can be regarded as a definition of an "elementary particle" (Jauch, 1968). The extension of the concept of localizability of elementary particles to relativistic quantum mechanics and field theory, and the construction of the corresponding *observable* (von Neumann, 1932)—the position operator—has been a topic for fundamental research since the original paper by T. D. Newton and E. P. Wigner in 1949 on "Localized States for Elementary Systems." In this work we have no intention of discussing the historical evolution of the concept of localizability (for a historical review of the problem of localizability, see Kalnay, 1971), but instead we shall discuss the concept in connection with that of primitive *causality.* In doing this we shall state a recent theorem due to G. C. Hegerfeldt (1974) which, essentially, says the *localization within a finite space(-time) region of a relativistic particle, in the sense described above, is inconsistent with causality.* We shall give another type of proof of the Hegerfeldt theorem and also mention how it is possible to arrive at the same result in the more rigorous formalism of the localization concept, as developed by J. M. Jauch & C. Piron (1967) and A. S. Wightman (1962). We remark, in passing, that the nonexistence of a covariant localization concept already was mentioned in the paper by A. S. Wightman cited above.

2. *Localization of a Nonrelativistic Particle*

Before entering into the relativistic consideration we shall consider some arguments, initially discussed by D. I. Blokhintsev (1968) some time ago, concerning the quantum mechanical description of a particle propagating in, e.g., a bubble chamber. When one is considering the very beautiful pictures from the current bubble chamber experiments (Kalmus, 1973), in high energy particle physics, one cannot avoid the question concerning the localization of the corresponding particle and its relation to the bubble chamber trajectory. Indeed, it is a *fundamental assumption* that the trajectory is associated with the particle in such a way that a kinematic analysis is possible (i.e., the trajectory is associated with a relativistic particle). In the experiments mentioned, the tracks are, of course, due to some complicated ionization

[1] In classical mechanics the concept of localizability is hidden in the most elementary concepts—this could be a reason why there does not exist any detailed analysis at all of this concept in classical physics.

process, but let us now extract the idea of a particle having a certain trajectory which also generates the corresponding bubble chamber trajectory (if the particle is charged). So we assume that we have a sequence of space-time points like that illustrated in Figure 1, which we now also identify with a bubble chamber trajectory. Each such space-time point (x_k, t_k), where k belongs to some appropiate index set, is associated with a localization of the propagating particle within a linear dimension of the order Δx_k (a quantity which we will make tend to zero). This means that at time $t = t_k$ the particle is localized within a space region V_k defined by

$$V_k \equiv \{x \mid x_k - \tfrac{1}{2}\Delta x_k \leqq x \leqq x_k + \tfrac{1}{2}\Delta x_k\} \tag{2.1}$$

Let us now investigate what quantum mechanics, in Feynman's path integral formulation (Feynman, 1949; see also Feynman & Hibbs, 1965, for a more detailed account), says about a trajectory such as that in Figure 1. Formally,

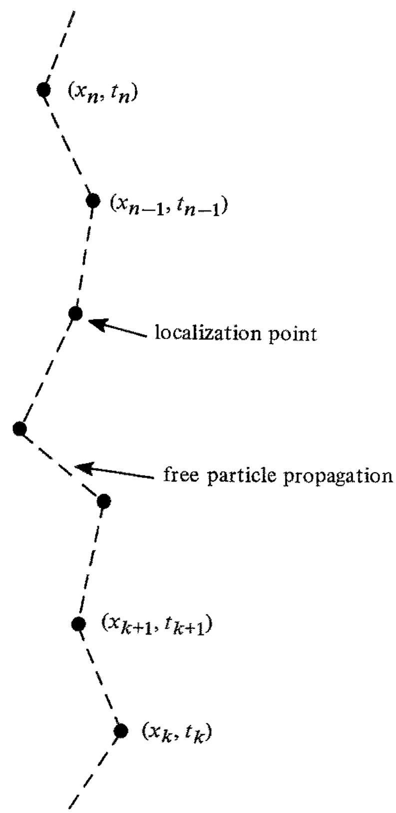

Figure 1–A model of a bubble chamber trajectory.

the probability $P(x_0, t_0; x_1, t_1; \ldots; x_n, t_n)$ for such a trajectory can be computed in the following way:

$$P(x_0, t_0; x_1, t_1; \ldots; x_n, t_n) = P(x_n, t_n; x_{n-1}, t_{n-1})P(x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}) \cdots P(x_1, t_1; x_0, t_0)\, \Delta x_1 \Delta x_2 \cdots \Delta x_{n-1} \tag{2.2}$$

where the conditional probabilities $P(x_m, t_m; x_{m-1}, t_{m-1})$ are given by

$$P(x_m, t_m; x_{m-1}, t_{m-1}) = G(x_m, t_m; x_{m-1}, t_{m-1})G^*(x_m, t_m; x_{m-1}, t_{m-1}) \tag{2.3}$$

In this expression $G(x_m, t_m; x_{m-1}, t_{m-1})$ is the causal Green's function for the free Schrödinger equation, i.e.,

$$G(x, t; x_{m-1}, t) = \delta(x - x_{m-1}) \tag{2.4}$$

and

$$G(x, t; x_{m-1}, t_{m-1}) = 0 \qquad \text{if } t < t_{m-1} \tag{2.5}$$

We now also assume that the particle is completely free between the localization points (x_k, t_k). We then obtain the following state function for (x, t) parameters such that $x_k < x < x_{k+1}$ and $t_k < t < t_{k+1}$:

$$G(x, t; x_k, t_k) = \frac{1}{N} \exp\left[i \frac{m}{2\hbar} \frac{(x - x_k)^2}{t - t_k} \right] \tag{2.6}$$

such that

$$i\hbar \frac{\partial G(x, t; x_k, t_k)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 G(x, t; x_k, t_k)}{\partial x^2} \tag{2.7}$$

and where N is a normalization which will be specified below. In order to find the mean energy of the "Markov chain" as described in Figure 1, we consider one fixed localization point, x_k, and let the point x_{k+1} tend to infinity (we consider, for simplicity, a one-dimensional propagation). We then notice that the state function (2.6) is nonintegrable [in the $L^2(R)$ sense] and therefore we introduce a regularization G_δ of G by performing an analytic continuation in the mass parameter, i.e., $m \to m + i\delta$:

$$G_\delta(x, t; x_k, t_k) \equiv \frac{1}{N_\delta} \exp\left[i \frac{m + i\delta}{2\hbar} \frac{(x - x_k)^2}{t - t_k} \right] \tag{2.8}$$

where we now introduce

$$N_\delta = \left[\frac{2\pi i\hbar(t - t_k)}{m + i\delta} \right]^{1/2} \tag{2.9}$$

i.e.,

$$N_\delta^* N_\delta = \left[\frac{(2\pi m\hbar\Delta t)^2 + (2\pi\delta\Delta t\hbar)^2}{(m^2 + \delta^2)^2} \right]^{1/2} \tag{2.9'}$$

which is perfectly regular when we take the limit $\delta \to 0$. This choice of the normalization constant corresponds to the criteria

$$\lim_{\delta \to 0} \lim_{\Delta t \to 0} G_\delta(x, t; x_k, t_k) = \delta(x - x_k) \tag{2.10}$$

where

$$\Delta t \equiv t - t_k \tag{2.11}$$

With this choice of normalization we know (Feynman, 1949; Feynmann & Hibbs, 1965) that the function G_δ above is a solution of the Schrödinger equation (2.5), under the condition that Δt is sufficiently small. This means that we can calculate the mean energy of the particle by using the following standard formula in quantum mechanics:

$$\langle E \rangle_\delta = \int_{x_k}^{\infty} G_\delta^*(x, t; x_k, t_k)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) G_\delta(x, t; x_k, t_k)\, dx \Big/ \int_{x_k}^{\infty} G_\delta^*(x, t; x_k, t_k) G_\delta(x, t; x_k, t_k)\, dx \tag{2.12}$$

Elementary calculations, using Gaussian integrals (see Appendix A), give us then the following result

$$\langle E \rangle_\delta = (\hbar/4m\Delta t)\,(m^2/\delta + \delta) \tag{2.13}$$

i.e.,

$$\langle E \rangle_\delta \to \hbar m/4\delta\Delta t \qquad \text{when } \delta \to 0 \tag{2.14}$$

Hence we see that the mean energy of the particle will be infinite if we have a pointlike localization at one space-time point. It turns out that the same argument for a finite number of localization points gives the same type of divergent mean energy (Appendix B). This result has also been obtained by D. I. Blokhintsev (1968) some time ago. It is interesting to notice that for an arbitrary number of localization regions, consisting of finite intervals, δ-function type of singularities cannot be avoided (Appendix C), i.e., the localization regions must contain "smooth tails."

For a finite number of localization points we must then, in general, introduce a "smearing" of the $G(x, t; x_k, t_k)$ function, and let us now study, for the sake of completeness, a simple example. We shall consider the effect of a trivial smearing, at the time $t = t_k$, with a Gaussian distribution. The state function at times $t > t_k$ can then be calculated according to the formula (Feynman, 1949; Feynman & Hibbs, 1965)

$$\psi(x, t) = \int_{-\infty}^{\infty} dx' N' \exp(-x'^2/2a^2) G(x, t; x', t_k) \tag{2.15}$$

where N' is a normalization constant such that

$$\int_{-\infty}^{\infty} dx\, \Psi(x,t)\, \Psi^*(x,t) = 1 \qquad \forall t \geqq t_k \tag{2.16}$$

Since all integrands in the expression for the mean energy are of a Gaussian type, we can find after some elementary calculations that

$$\langle E \rangle = (\hbar^2/2m)\,(1/2a^2) \tag{2.17}$$

which is in no way surprising and, of course, a well-known result in quantum mechanics.

3. *Localization and Causality*

By using heuristic arguments we have shown that a pointlike localization of an elementary particle can introduce a divergent mean energy. We noticed, furthermore, that a "smearing" of the free particle state function changed this fact drastically. We showed this explicitly for the case when the smearing function had its support on the whole of the real line. A finite energy will also, of course, emerge in the case with a smearing function which has a compact support, if the corresponding function is sufficiently smooth (e.g., C^∞ functions with compact support).

Going over to the relativistic domain, we now ask ourselves the question if it is possible to localize a particle in a finite space(-time) region. We shall see below that this is *impossible* if we insist that the relativistic theory should be consistent with a causality condition. This is the so-called Hegerfeldt (1974) theorem, and in order to derive this result we need some definitions. We introduce first of all a definition concerning the meaning of the concept of localization of a particle within a finite space region.

> *Definition I:* A one-particle state, ϕ_{x_0}, is said to be *localized in a space volume V_{x_0} at the time x_0*, if the probability of finding the particle in V_{x_0} is unity. It is said to *be not in V_{x_0} at the time x_0* if the corresponding probability of finding the particle in V_{x_0} is zero.

By assumption we then have an operator $E(V_{x_0})$ associated with the volume V_{x_0} in such a way that its expectation value in the state ϕ_{x_0} gives the probability $p(V_{x_0})$ of finding the particle in the corresponding volume, i.e.,

$$p(V_{x_0}) = (\phi_{x_0}, E(V_{x_0})\, \phi_{x_0}) \tag{3.1}$$

We now assume (i) that the states of the particle under consideration transform according to an *irreducible representation of the Poincaré group*, and (ii) a *spectral condition* i.e., the generator of time translations has a spectrum which is bounded from below.

Let us consider the localized one-particle state ϕ_{x_0} at a later time $x_0' > x_0$. The corresponding localization volume will be denoted by the symbol $V_{x_0'}$. Since the particle necessarily has a finite propagation velocity ($< c$) the localization volume $V_{x'_0}$ must be contained in the causal future cone of V_{x_0}, as illustrated in Figure 2. We now perform a spacelike transformation of the localized one-particle state $\phi_{x'_0}$ to the localized state $\phi_{x'_0}$, which describes a particle localized in a volume $V_{x'_0}$. We assume that the volumes $V_{x'_0}$ and $V_{x'_0}$ are totally spacelike separated, as illustrated in Figure 2. A notation of *causality* can then be introduced according to (Hegerfeldt, 1974)

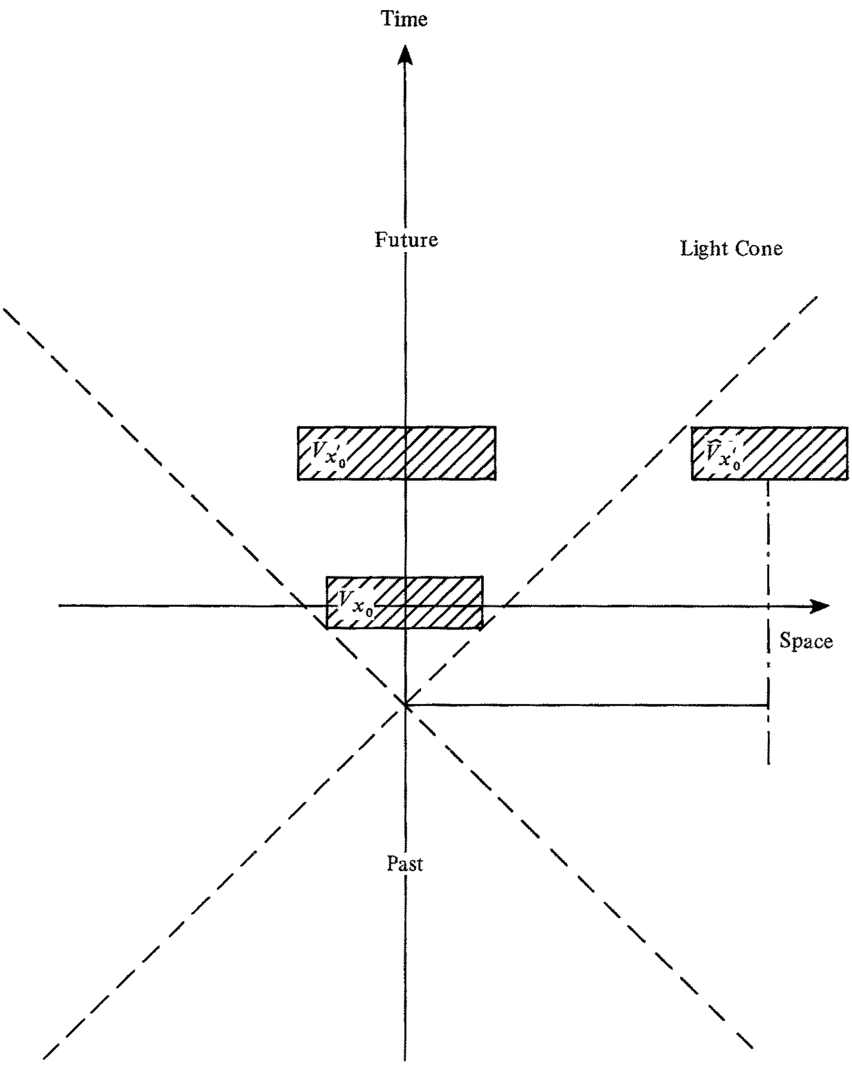

Figure 2–Supports of the localized one particle states.

Definition II: Suppose that a one-particle state at time x_0, ϕ_{x_0}, is localized in a volume V_{x_0}. Then there exists a spacelike vector $\mathbf{x}'$ such that at the time $x_0' > x_0$ the particle, when translated with the vector $\mathbf{x}'$, is not in the volume $V_{x_0'}$.

It is then a remarkable fact that localization of a particle in a relativistic theory is incompatible with this concept of causality. Indeed we have the following theorem due to Hegerfeldt (1974) which we are going to prove in somewhat different setting:

Theorem: In a relativistic theory of particles, there is no one-particle state localized in a finite space region[2] satisfying the causality condition in Definition II.

Proof: Let us introduce the following four-vector $\tilde{x}_\mu$ with components

$$\tilde{\mathbf{x}}_k = \mathbf{x}_k' \qquad k \in \{1, 2, 3\}$$

$$\tilde{x}_0 = x_0'$$

Using the transformation properties of the one-particle states under space and time translations we have that

$$\hat{\phi}^s_{x_0'} = U(\tilde{x}, \mathbb{1})\, \phi^s_{x_0} \tag{3.2}$$

where we suppress a summation over spin or helicity indicies (we use the notation defined in Streater & Wightman, 1964). From the assumption of causality it now follows that the relativistic invariant scalar product between the one-particle states $\hat{\phi}_{x_0'}$ and ϕ_{x_0} is zero, i.e.,

$$\begin{aligned}(\phi_{x_0}, \hat{\phi}_{x_0'}) &= \int d\Omega_m(p) < \phi^s_{x_0} \,|\, \mathbf{p} \times \mathbf{p} \,|\, | U(\tilde{x}, \mathbb{1}) \,|\, \phi^s_{x_0} > \\ &= \int d\Omega_m(p) \exp(ip_\mu \tilde{x}^\mu)\, |\phi^s_{x_0}(p)|^2 = 0 \end{aligned}\tag{3.3}$$

Here we have introduced a relativistic invariant measure on the one-particle mass hyperboloid (see Appendix D, where we give some calculations for the case $s = 0$). Let us now define the following holomorphic function $I(.)$:

$$I(z) \equiv \int d\Omega_m(p) \exp(ip_\mu z^\mu)\, |\phi_{x_0}(p)|^2 \tag{3.4}$$

where

$$z \in \tau = \{z \mid z = \tilde{x} + iy, \text{ where } \tilde{x} \text{ is spacelike and } y \in V_+, \text{ the positive forward lightcone}\}$$

From the spectral condition we see that

$$I(z) < \infty \qquad \forall z \in \tau \tag{3.5}$$

2 From the proof it will be clear that the theorem is valid also for a finite space-time region.

and, moreover, we have by the causality assumption the following boundary value:

$$\lim_{\operatorname{Im} z \to 0} I(z) = \int d\Omega_m(p) \exp(ip_\mu \tilde{x}^\mu) \, | \phi_{x_0}^s(p) |^2 = 0 \tag{3.6}$$

By construction we thus have a holomorphic function $I(.)$ defined on a set

$$\mathscr{B} \equiv (R^4 + i\mathscr{C}) \cap \mathscr{O}$$

where $\mathscr{C}$ is a convex cone and the open set $\mathscr{O} \subset C^4$ contains a real enviroment E such that

$$\lim_{y \to 0} I(x + iy) = 0 \qquad \forall\, x \in E \tag{3.7}$$

A generalization of the classical edge of the wedge theorem[3] then implies that

$$I(z) = 0 \qquad \forall\, z \in \mathscr{B} \tag{3.8}$$

and from this we conclude that

$$(\phi_{x_0}, \phi_{x_0'}) = 0 \qquad \forall\, x_0' \tag{3.9}$$

From this equation we, trivially, realize that

$$\phi_{x_0} = 0$$

and hence there does not exist any nontrivial relativistic one-particle state that is localized in a finite space(-time) volume under the assumption of causality, as defined above.

4. *Systems of Imprimitivities and Localization of Elementary Particles*

In the last section we introduced an operator $E(V)$ (we suppress the time dependence) such that the expectation defined by equation (3.1) gives the probability of finding the particle, corresponding to the one-particle state ϕ, in the volume V. If this probability is equal to unity we see, by using the Cauchy-Schwartz inequality (Yosida, 1974), that

$$E(V)\, \phi = \phi \tag{4.1}$$

i.e., $E(V)$ becomes a projection operator.

In the Jauch-Piron approach to quantum mechanics (Jauch, 1968; Piron, 1972), the set of *propositions* of a physical system constitute a complete orthocomplemented lattice. Under very general conditions this propositional system can be represented (Piron, 1964) by the lattice of all closed subspaces of a Hilbert space, i.e., by projectors. Localization of an elementary particle then corresponds to a certain structure of the proposition system. Indeed, the proposition system must contain, almost trivially, propositions that answer the question whether the particle *is* or *is not* in a certain region in space.

[3] Theorem 2-17 in Streater & Wightman (1964).

In the Wightman (1962) and Jauch-Piron (1967) approach to the question of localizability of an elementary particle, we have a proposition associated with each space volume corresponding to the particle localization volume (we consider the set of propositions at a fixed time and we will work in the Heisenberg picture). More precisely we assume that (Wightman, 1962)

(i) For every Borel set, $V \subset R^3$, there exists a projection $E(V) \in B(\mathscr{H})$, i.e., a bounded linear operator on the Hilbert space, $\mathscr{H}$, of physical states. The expectation value of $E(V)$ gives the probability of finding the particle in the volume V.

(ii) For every pair of Borel sets V_1 and $V_2 \subset R^3$ we have that

$$E(V_1 \cap V_2) = E(V_1)E(V_2) \; [= E(V_2)E(V_1)] \tag{4.2}$$

i.e., compatible propositions for all Borel sets.

(iii) For every pair of Borel sets V_1 and $V_2 \subset R^3$ we have that

$$E(V_1 \cup V_2) = E(V_1) + E(V_2) - E(V_1 \cap V_2) \tag{4.3}$$

which corresponds to the statement that the set of states which are localized in the volume $V_1 \cup V_2$ is the closed linear manifold spanned by localized states in the volumes V_1 and V_2, respectively.

(iv) We have $E(R^3) = 1$ (the identity operator) (4.4)

i.e., the probability must be unity of finding the particle somewhere in the configuration space.

(v) For every Borel set $V \subset R^3$ we have that

$$E(RV + \mathbf{a}) = U(\mathbf{a}, R)E(V)U^{-1}(\mathbf{a}, R) \tag{4.5}$$

which gives the transformation properties of the projectors under the Euclidean group of Motions. R is a rotation and the vector $\mathbf{a}$ corresponds to a translation. $U(\mathbf{a}, R)$ is the corresponding unitary operator whose action on the physical states induces a rotation R and a translation with the vector $\mathbf{a}$.

In the sense of Mackey (1968) the formal structure (i)-(v) defines *a system of imprimitivity* for the representation $U(\mathbf{a}, R)$ of the Euclidean group with *base* R^3.

The importance of this formulation of localization rests on the fact that, if we make the following definition of an *elementary particle* (Jauch, 1968):

Definition III: A localizable system with the system of imprimitivity

$$\{E(V), U(\mathbf{a}, R)\}$$

describes an *elementary particle* if the system of imprimitivity is irreducible, i.e., the commutant

$$\{E(V), U(\mathbf{a}, R)\}'$$

is a multiple of the identity operator,

then the so-called *imprimitivity theorem of Mackey* gives a complete classification of all "elementary particles" and thereby a determination of all localizable systems in nature.

The mapping $V \to E(V)$ with the properties (i)-(v) also defines a *spectral measure* and hence we also have, by the spectral theorem (Yosida, 1974), a position operator. It now turns out that the photon is not localizable in this sense (Wightman, 1962) but the formal structure (i)-(v) can be generalized in the following sense (Jauch & Piron, 1967):

(i′) For every Borel set $V \subset R^3$ we associate a projection $E(V)$ in the same way as in (i).

(ii′) $$E(R^3) = 1 \quad \text{and} \quad E(\Phi) = 0 \tag{4.6}$$

where Φ denotes the empty set.

(iii′) For every pair of disjoint sets V_1 and $V_2 \subset R^3$ we have that

$$E(V_1) \perp E(V_2) \tag{4.7}$$

where the symbol $\perp$ denotes disjoint projectors (the corresponding closed subspaces of the Hilbert space $\mathscr{H}$ are disjoint),

(iv′) For every pair of Borel sets V_1 and $V_2 \subset R^3$ the following relations holds:

$$E(V_1 \cap V_2) = \underset{n \to \infty}{\text{strong limit}}\ [E(V_1)E(V_2)]^n \tag{4.8}$$

(v′) As before.

This generalization has been constructed in order to describe localization of massless particles with spin $s \neq 0$ (e.g., the photon) (Amrein, 1969), and it defines a *generalized system of imprimitivities.* In this case we do not have a relation of the type defined by equation (4.2), i.e., the projectors are not in general commutative. A sufficient and necessary condition can now be given which reduces the system (i′)-(v′) to the system (i)-(v) (Jauch & Piron, 1967):

$$E(V) + E(V - R^3) = 1 \qquad \forall \text{ Borel set } V \subset R^3 \tag{4.9}$$

We shall now see that it is possible to give another type of proof of the Hegerfeldt theorem, discussed above, in this abstract setting of localizability of elementary particles.

As in Figure 2 we consider the localization of the particle on a hyperplane with fixed time x_0. Furthermore we consider the propositions corresponding to (i) the particle being localized in the volume V_{x_0} at time x_0 and (ii) the particle being in the volume $V_{x_0'}$, the spacelike translated volume of V_{x_0}, as before, where $x_0' > x_0$. We denote the corresponding projectors by the symbols $E(V_{x_0})$ and $E(\hat{V}_{x_0'})$. According to the assumption of causality we have that these projectors are disjoint, i.e.,

$$E(V_{x_0}) \perp E(\hat{V}_{x_0'}) \tag{4.10}$$

Moreover, they commute because the volumes are spacelike separated.

Let us now *assume* that the time evolution can be unitarily implemented in the sense that

$$E_{x_0'} \equiv E(V_{x_0'}) = U(x_0')EU^{-1}(x_0') \tag{4.11}$$

where $E \equiv E(V_{x_0})$ and where we, without loss of generality, have put $x_0 = 0$. Let us also introduce the notation $F_{x_0} \equiv E(\hat{V}_{x_0})$ and we see that

$$EF_{x_0'} = F_{x_0'}E \tag{4.12}$$

which is true for $|x_0'| < 1$, if the $|\mathrm{x}'|$ is sufficiently large. To proceed further we now notice the following theorem[4] due to H. J. Borchers (1967).

Theorem II: Assume we have a continuous representation

$$R \ni t \to U(t)$$

of a one-parameter group with a semibounded spectrum. Moreover assume that we have two projectors E, F such that

$$U(t)FU^{-1}(t)E = EU(t)FU^{-1}(t) \qquad \forall\, t \text{ such that } |t| < 1.$$

If we have that $FE = 0$ then it follows that

$$U(t)FU^{-1}(t)E = 0 \qquad \forall\, t$$

Since we assume that we have a spectral condition we see that the causality assumption implies that equation (40) is valid for all times x_0' ($x_0 = 0$). We now assume that there exists a nontrivial vector, ϕ_0, which describes a particle localized in the volume V_{x_0} at time x_0. This means that equation (4.1) is valid. Hence we see that

$$\begin{aligned}
&(\phi_0, EU(x_0')U(\mathrm{x}')EU^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0)\\
&= (\phi_0, U(x_0')U(\mathrm{x}')EU^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0)\\
&= (U^{-1}(x_0')U^{-1}(\mathrm{x}')\phi_0, EU^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0) = 0
\end{aligned} \tag{4.13}$$

by construction. Since E is a positive and self-adjoint we derive the following equation:

$$EU^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0 = 0 \tag{4.14}$$

and from this we have that

$$(\phi_0, EU^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0) = 0 \tag{4.15}$$

Hence we have proved that

$$(\phi_0, U^{-1}(\mathrm{x}')U^{-1}(x_0')\phi_0) = 0 \qquad \forall\, x_0' \text{ and } |\mathrm{x}'| \text{ sufficiently large} \tag{4.16}$$

[4] This proof technique, by using a theorem of Borchers, was suggested to the author by B. Jamewicz at a seminar in Göteborg 1975.

Explicitely equation (4.16) means that

$$\int d\Omega_m(p)\,|\phi_0{}^s(p)|^2 \exp(i\mathbf{p}\cdot\mathbf{x}' - i(\mathbf{p}^2 + m^2)^{1/2}x_0') = 0 \qquad \forall x_0' \qquad (4.17)$$

But this equation leads to a contradiction since we now can show that $\phi_0 = 0$ either by using relativistic invariance arguments or Payley-Wiener type of theorems (Yosida, 1974). Hence we have once more proved that localization of a relativistic particle in a finite volume is inconsistent with the concept of causality as defined above.

5. *Conclusions*

By using a naive and unrealistic model for a particle propagating in a bubble chamber (Figure 1), we showed that a pointlike localization leads to a divergent energy. This is true also in the case when we have localization regions with sharp boundaries as explained in Appendix C. These results are of course in complete agreement with the Heisenberg uncertainty relation $\Delta x \cdot \Delta p \cong \hbar$. The example also illustrates the relevance of a model concerning measurements in the quantum domain (the reduction of wave packet is an essential step in constructing the state function defined by equation C4). In a forthcoming work we shall make a detailed analysis of this problem in the context of quantum Markov processes.

The question of localization of elementary particles was then extended to the relativistic domain and we gave an elementary proof of the Hegerfeldt theorem (in the Schrödinger picture) saying that localization within a finite space(-time) region of a single particle is inconsistent with causality. The proof technique was based on the same type of arguments as used in the proof of the well-known Reeh-Schlieder theorem from 1961 (Streater & Wightman, 1964).

In the last section we formulated the concept of localization in the framework of systems of imprimitivities—a specific structure on the system of propositions of the physical system under consideration. The use of a remark by Borchers was the essential step in proving the Hegerfeldt theorem in this setting (Heisenberg picture).

The results of this paper confirms a statement due to Wightman (1962) some time ago, namely, that ". . . a sensible notion of localizability in space-time does not exist." This could be interpreted in the sense that the notion of configuration has no consistent meaning in the microworld (Kalnag, 1973) as far as relativistic theories are concerned. Hence it could also indicate that an extension of stochastic processes on a classical configuration space (Skagerstam, 1975) to the relativistic domain could be impossible. In any case the problem calls for further analysis.

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Appendix A

In order to calculate the expression defined by equation (2.12) we introduce the following notation:

$$\alpha_k \equiv \frac{i(m + i\delta)}{2\hbar(t - t_k)} \tag{A1}$$

$$\beta_k \equiv \frac{\delta}{\hbar(t - t_k)} \tag{A2}$$

and

$$\gamma_k \equiv (N_\delta^* N_\delta)^{-1} \tag{A3}$$

After some elementary transformations we see that on the one hand

$$\int_{x_k}^{\infty} G_\delta^*(x', t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) G_\delta(x', t)\, dx'$$

$$= -\frac{\hbar^2}{2m} - \left\{2\alpha_k\gamma_k \int_{x_k}^{\infty} \exp[-\beta_k(x' - x_k)^2]\, dx'\right.$$

$$\left. + 4\alpha_k{}^2\gamma_k \int_{x_k}^{\infty} (x' - x_k)^2 \exp[-_k(x' - x_k)^2]\, dx'\right\}$$

$$= -\frac{\hbar^2}{2m}\frac{\sqrt{\pi}}{2}\frac{2\alpha_k\gamma_k}{\beta_k^{1/2}} + \frac{2\alpha_k{}^2\gamma_k}{\beta_k^{3/2}} \tag{A4}$$

and on the other

$$\int_{x_k}^{\infty} G_\delta^*(x', t) G_\delta(x', t)\, dx' = \frac{\gamma_k}{\beta_k^{1/2}}\frac{\sqrt{\pi}}{2} \tag{A5}$$

where we have used the notation $G(x', t) = G(x, t; x', t_k)$. Straightforward calculations then gives us the expression (2.13).

Appendix B

In Appendix A we discussed the mean energy for a particle performing a motion in the configuration space in the way illustrated in Figure 1. We showed that the corresponding mean value of the energy operator was divergent, when we had one localization point. In this Appendix we shall now investigate the same problem in the case of an arbitrary finite number, say N, of localization points. What we have to do in order to find the mean energy for this case is, essentially, to change the integration domain of the integrals in the expression (2.12) in the following way:

$$\int_{x_k}^{\infty} \to \sum_{m=k}^{N-1} \int_{x_m}^{x_{m+1}} + \int_{x_N}^{\infty} \tag{B1}$$

Since we have the following estimate:

$$\sum_n \int_{x_n}^{x_{n+1}} \frac{m}{2\pi\hbar(t-t_n)} \exp\left(-\frac{\delta}{\hbar}\frac{(x'-x_n)^2}{(t-t_n)}\right) dx'$$

$$\leqq \frac{m}{2\pi(\hbar\Delta t\delta)^{1/2}} N \int_0^{\hat{\beta}} \exp(-x'^2)\, dx' < \infty \qquad \forall\, \delta \geqq 0 \tag{B2}$$

where

$$\Delta t = \min_{0 \leqq n \leqq N} (t - t_n) \tag{B3}$$

and

$$\hat{\beta} = \max_{0 \leqq n \leqq N} \left[\frac{\delta}{\hbar(t-t_n)}\right]^{1/2} \Delta x_n \tag{B4}$$

and a similar one for the expectation value

$$\sum_n \int_{x_n}^{x_{n+1}} G_\delta^*(x', t) G_\delta(x', t)\, dx' < \infty \qquad \forall\, \delta \geqq 0 \tag{B5}$$

we see that the same type of divergence will appear as in the case with one localization point, because of the last integration in (B1).

Appendix C

In this Appendix we shall consider the path described in Figure 1 in the case when we have an arbitrary number of localization points. In order to describe the corresponding state function we introduce the following characteristic function $\chi_{[t_k, t_{k+1}[}(t)$, which is such that

$$\chi_{[t_k, t_{k+1}[}(t) = \begin{cases} 1 \text{ iff } t \in [t_k, t_{k+1}[\\ 0 \text{ otherwise} \end{cases} \tag{C1}$$

If $H(t)$ is the Heaveside step function, i.e.,

$$H(t) = \begin{cases} 1 \text{ iff } t \geqq 0 \\ 0 \text{ otherwise} \end{cases} \tag{C2}$$

we can represent the characteristic function χ, described above, in the following way:

$$\chi_{[t_k, t_{k+1}[}(t) = H(t - t_k) - H(t - t_{k+1}) \tag{C3}$$

The state function for the inifinitely extended path can then be written in the following form:

$$\Psi_\delta(x, t) = \sum_k G_\delta(x, t; x_k, t_k)\chi_{[t_k, t_{k+1}[}(t)\chi_{[x_k, x_{k+1}[}(x) \tag{C4}$$

where we use the regularized $G_\delta(x, t; x_k, t_k)$ as defined by equation (2.8). In order to find the mean energy we have to calculate

$$\langle E\rangle_\delta = \int_{-\infty}^{\infty} \Psi_\delta^*(x', t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Psi_\delta(x', t)\,dx' \int_{-\infty}^{\infty} \Psi_\delta^*(x', t)\Psi_\delta(x', t)\,dx' \tag{C5}$$

Elementary calculations then gives us that[5]

$$\begin{aligned}\int_{x_k}^{x_{k+1}} \Psi_\delta^*(x', t)\frac{\partial^2}{\partial x^2}\Psi_\delta(x', t)\,dx' = {} & \frac{2\alpha_k\gamma_k}{\beta_k^{1/2}}\{(\alpha_k/\beta_k)\ [I(\beta_k^{1/2}\Delta x_k) \\ & - \beta_k^{1/2}\Delta x_k \exp(-\beta_k\Delta x_k)] + I(\beta_k^{1/2}\Delta x_k)\} - 2\alpha_k\Delta x_k \\ & \times \Psi_\delta^*(x_{k+1}, t)\Psi_\delta(x_{k+1}, t) - \tfrac{1}{2}\{[(\partial/\partial x)\,|\,\Psi_\delta(x, t)\,|^2]_{x=x_k} \\ & - [(\partial/\partial x)\,|\,\Psi_\delta(x, t)\,|^2]_{x=x_{k+1}} + \delta(0)[\,|\,\Psi_\delta(x_{k+1}, t)\,|^2 + |\,\Psi_\delta(x_k, t)\,|^2]\}\end{aligned} \tag{C6}$$

where

$$I(x) \equiv \int_0^x \exp(-x'^2)\,dx' \tag{C7}$$

In the same manner we have that

$$\int_{x_k}^{x_{k+1}} \Psi_\delta^*(x', t)\Psi_\delta(x', t)\,dx' = \frac{\gamma_k}{\beta_k^{1/2}}\,I(\beta_k^{1/2}\Delta x_k) \tag{C8}$$

Here we notice that in both equations (C6) and (C8) we have restricted the time parameter to the interval $[t_k, t_{k+1}[$, otherwise we have no contribution

5 Here we use the notation $\Delta x_k = x_{k+1} - x_k$.

at all. Collecting the result and performing the appropiate limit procedures we arrive at the following result:

$$\lim_{\delta \to 0} \langle E \rangle_\delta = \lim_{\delta \to 0} \left\{ \frac{-\sum_k (\hbar^2/2m)\, \frac{2}{3}\alpha_k^2 \gamma_k (\Delta x_k)^3 \chi_{[t_k, t_{k+1}[}(t)}{\sum_k \gamma_k \Delta x_k \chi_{[t_k, t_{k+1}[}(t)} + \frac{\hbar^2}{2m} \frac{\sum_k \delta(0) \gamma_k \chi_{[t_k, t_{k+1}[}(t)}{\sum_k \gamma_k \Delta x_k \chi_{[t_k, t_{k+1}[}(t)} \right\} \tag{C9}$$

For the case when $t \in [t_k, t_{k+1}[$ we derive the following expression:

$$\lim_{\delta \to 0} \langle E \rangle_\delta = \tfrac{1}{2} \cdot \tfrac{1}{6} m \left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \frac{\hbar^2}{2m} \delta(0) \frac{1}{\Delta x_k} \tag{C10}$$

Hence we see that the δ-function type of singularities cannot be avoided. Since they arise because of the sharp localization region boundaries we must, in order to have a finite mean energy, in general have a "smooth tail" in the localization regions.

Appendix D

In this Appendix we shall discuss the relativistic invariant scalar product in equation (3.3). Here we make use of a continuum normalization (in the Dirac bra-ket notation)

$$\langle \mathbf{p} \mid \mathbf{p}' \rangle = \delta^3(\mathbf{p} - \mathbf{p}') p^0 \tag{D1}$$

where

$$p^0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{D2}$$

The scalar product between two one-particle states is then defined by the following expression:

$$(\phi, \psi) \equiv \int d\Omega_m(p) \phi^*(\mathbf{p}) \psi(\mathbf{p}) \tag{D3}$$

where $d\Omega_m(p)$ is a relativistic invariant measure on the one-particle mass hyperboloid $p_\mu p^\mu = m^2$ (here we use a timelike metric as in Streater & Wightman, 1964):

$$d\Omega_m(p) \equiv d^3\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} \tag{D4}$$

Let us now consider the special case of a scalar particle ($s = 0$). It $\{a, \Delta\} \in \mathscr{P}_+^\uparrow$ then (Streater & Wightman, 1964)

$$[U(a, \Delta)\phi](p) = \exp(ip_\mu a^\mu)\phi(\Delta^{-1} p) \tag{D5}$$

and hence we see that

$$(\phi_{x_0}, \hat{\phi}_{x'_0}) = \int d\Omega_m(p) \langle \phi_{x_0} | \mathbf{p} \rangle \langle \mathbf{p} | U(x, \mathbb{1}) | \phi_{x_0} \rangle$$

$$= \int\int d\Omega_m(p) d\Omega_m(p') \langle \phi_{x_0} | \mathbf{p} \rangle \exp(ip'_\mu x^\mu) \langle \mathbf{p} | \mathbf{p}' \rangle \langle \mathbf{p}' | \phi_{x_0} \rangle$$

$$= \int d\Omega_m(p) \exp(ip_\mu x^\mu) | \phi_{x_0}(p) |^2$$

In the case when $s \neq 0$ one obtains a sum over spin or helicity indicies, a com· lication which we disregard in this paper.

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